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Dear Sir,

You will find below

(A) Tables containing the number of elements of each possible order in the group of the cube, as well as in the group of the edges and the group of the corners (together with the formulae used and their derivation);

(B) A few minor corrections to pp. 50-51 of your 'Notes on Rubik's Magic Cube'.

In order to determine the distribution of elements over the possible orders in the group Z_k wr S_n, we organize the elements into 'families' according to their cycle structure (regardless of possible flips or twists). Since the sum of the cycle lengths in the typical form must be n, there are as many families in Z_k wr S_n as there are non-ordered partitions of the number n. Using well-known recurrence formulae (or tables) for the number p(n) of such partitions one finds that there are p(12)=77 families in the edge group G_e and p(8)=22 families in the corner group G_c.

Consider any such family. If g_i is the number of i-cycles in the typical form, the number of elements in the family is

$$N_f = \frac{n!k^n}{k\prod_i g_i!i^{g_i}}. \qquad (1)$$

The numerator is the number of ways to write the pieces (edges or corners) into the typical form with arbitrary orientations. In the denominator, the factorials express the number of ways to permute cycles of equal length among themselves; the cycle-length powers, the number of equivalent orders in which to write the pieces within each cycle; the factor k in front of the product, the restriction to orientation-conserving forms. In addition to the factors exhibited above, the numerator should contain a factor k for each cycle (the number of possible signs) and the denominator a factor k, cancelling the corresponding factor in the numerator, for each cycle (the number of ways to change the orientation of all pieces in the cycle as a whole).

To take an example, the edge-group family in which the typical form consists of one 5-cycle, two 2-cycles, and three 1-cycles, comprises

$$\frac{12!2^{12}}{2\cdot 1!5^1 2!2^2 3!1^3} = 40874\ 80320$$

elements. To designate particular families, we have been using schemes like (5 2 2 1 1 1) for the family described above, containing the cycle lengths in numbers corresponding to their multiplicities g_i (we call such a scheme the 'characteristic' of the family).

Now there are just two possible orders of elements in any family, the one being the least common multiple of the cycle lengths represented in the typical form, the other being k times the former. In the above example from G_e, the lower order (10, the LCM of 5 and 2) results whenever the 2-cycles are unflipped. In the general case one must determine the highest power of k dividing any cycle length, then seek out the cycle lengths divisible by this leading power of k. If A be the class of such cycle lengths, the elements of the lower order are those in which all cycles of lengths belonging to A have zero total change of orientation (i.e. are unflipped or untwisted, as the case may be). In the family of n-cycles, only one order is possible since there can be no net orientation change.

If the class A introduced above does not comprise all cycle lengths occurring in the typical form, the number of elements of the lower order in a family is given by

$$N_d = \frac{N_f}{k^{\Sigma' g_i}}, \qquad (2)$$

the primed sum extending over the cycle lengths in A. This sum is the number of cycles for which the sign (viz., a zero) is prescribed, and so the denominator in (2) is the measure of the limitation imposed on the choice of signs in the typical form. – In the case where all cycle lengths belong to A, the correct expression for N_d is obtained by taking the right-hand side of (1), deleting the factor k in the denominator, and inserting in (2). The rationale for this procedure lies in the fact that the division by k was meant to express the effect of demanding conservation of total orientation, a condition that is now secured through the prescription of all cycle signs.

Reverting to our standard example: The 'leading power of k' among the divisors of the cycle lengths in the G_e-characteristic (5 2 2 1 1 1) is $k^1=2^1$; hence the class A contains a single cycle length, 2. The sum $\Sigma' g_i$, then, consists of only one term, namely $g_2=2$. Since the typical form contains cycle lengths other than the one in A, we can use expression (2) directly and find that the number of elements of order 10 in the family considered is

$$N_d = \frac{N_f}{k^2} = \frac{4087480320}{4} = 10218\ 70080.$$

By subtracting this from the total number of elements in the family, we find that there are 4087480320-1021870080 =3065610240 elements of the higher order (20) in the same family.

We have made lists of the families in G_e and in G_c, and (using a programmable pocket calculator) computed the number of elements of either order in each family. Next, we have obtained by summation the number of elements of every possible order, separated according to parity, in these groups. The results are recorded in Tables 1 and 2 in the appendix. Combining edge and corner elements in the manner outlined by yourself (like much of what we have presented above) in Section 5.10.D, we finally obtained Tables 3 and 4 showing the number of elements of each possible order in the whole group G_o. For the many multiplications, summations, and calculations of LCMs required in this last stage, we made use of a large computer.

For the possible interest it may have, we give here our results on a further classification of the groups G_e and G_c. Each family can be divided into classes distinguished by the distribution of flips or twists. Extending the use of characteristics from families to classes, we denote the twelve classes in the family (5 2 2 1 1 1) by (5 2 2 1 1 1), $(5_+2\ 2_+1\ 1\ 1)$, $(5_+2\ 2\ 1\ 1\ 1_+)$, $(5\ 2_+2_+1\ 1\ 1)$,...,$(5_+2_+2_+1\ 1\ 1_+)$, ...,$(5_+2_+2_+1_+1_+1_+)$. The subscripts are used in the same manner as in the standard notation for particular cycles.

How many classes does a given family contain?

Take g cycles of equal length, considered as indistinguishable prior to the attribution of signs. With a repertoire of k different signs, and taking no heed of the conservation laws for the moment, we can furnish the cycles with signs in

$$C_{g+k-1,k-1} = \frac{(g+k-1)!}{g!(k-1)!}$$

distinct ways. For shortness we write this number as $K_{g,k}$. Thus, the formulae needed in dealing with G_e and G_c are $K_{g,2} = g+1$, $K_{g,3} = \frac{1}{2}(g+1)(g+2)$. The number of distributions of signs compatible with the conservation law would seem to be just $\frac{1}{k}$ times the total number. Unfortunately, $K_{g,k}$ is divisible by k only when g is not. There seems to be no general expression for the number we are seeking. In the cases of k=2 or k=3, however, it can be seen with little effort that signs totalling to 0 (mod k) or to m (mod k) with $m \neq 0$ can be chosen in

$$\left\lceil \frac{K_{g,k}}{k} \right\rceil \quad \text{or} \quad \left\lfloor \frac{K_{g,k}}{k} \right\rfloor$$

ways, respectively. Here $\lceil\,\rceil$ and $\lfloor\,\rfloor$ denote the 'upper' and 'lower' integral parts. It can be shown that in the case of several cycle lengths, the number of different ways to distribute signs totalling to 0 (mod k) is given by

$$\left\lceil \frac{1}{k} \prod_i K_{g_i,k} \right\rceil .$$

For the family with $g_5=1$, $g_2=2$, and $g_1=3$ (all g's except these vanishing), this formula gives: $\left[\frac{1}{2}(1+1)(2+1)(3+1)\right] = 12$, in accordance with what one finds by writing out all possible schemes of signs for this case. Calculating in this way the number of classes in each family in the edge and corner groups, we have found that there are 590 classes in G_e, 281 in G_c. The number of elements in a class is given by

$$N_{cl} = \frac{n!k^n}{\prod_{i,s} g_{is}!(ik)^{g_{is}}},$$

where g_{is} is the number of i-cycles with the sign s (in G_e, s can take on the values 0 or +; in G_c, the values 0,+,-). Comparing with the expression for N_f, we note that factorials of g_i's have been replaced by factorials of g_{is}'s, since only cycles with the same sign (as well as the same length) may be permuted. The factors of k in the denominator owe their presence to the prescription of all signs. - For the class $(5_+ 2\ 2\ 1\ 1\ 1_+)$, we have $g_{5+}=1$, $g_{2o}=2$, $g_{1o}=2$, $g_{1+}=1$, and so the number of elements is

$$\frac{12!2^{12}}{1!(5\cdot 2)^1 2!(2\cdot 2)^2 1!(1\cdot 2)^1 2!(1\cdot 2)^2}$$
$$= 3832\ 01280.$$

It will be seen from the Tables below that our results for the case of the order m=3 differ somewhat from the numbers you have presented in Section 5.10.D of the Fifth Edition. For the number of corner elements of orders 3 or 1, we get 355994+1 =355995 (against 355975 in your Errata); for the total number of elements in the cube group of order 3, we get 33894540622394 (against 22455595008000; it seems that in the product of the corner element and edge element numbers, you have taken for the second factor the number 63078400 (the last term of the sum in line 29) instead of 95210721).

Concerning the first problem on p. 50 (possible orders in the group of the cube): You will see from our Tables that there are 73 different possible orders. Among the nontrivial nonorders, the first nonobvious case is $2^3 11 = 88$ (not 385). It is a nonorder since an element of this order would have to be an 11-cycle on edges combined with an 8-cycle on corners, which is impossible on account of its parity. The nonobviousness is manifest.

Very truly yours,

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Order	Even	Odd
1	1	–
2	80 80447	76 07424
3	952 10720	–
4	27625 27680	36643 66464
5	12265 48224	–
6	1 73874 74720	1 28558 28480
7	364 95360	–
8	3 58506 08640	1 88437 70880
9	2 04374 01600	–
10	3 41690 84928	3 03495 41376
11	4 45906 94400	–
12	5 59793 20320	12 30474 85440
14	55107 99360	58392 57600
15	14476 49280	–
16	3 06561 02400	1 02187 00800
18	3 40623 36000	2 72498 68800
20	3 98529 33120	3 75026 31936
21	29196 28800	–
22	4 45906 94400	–
24	2 46951 93600	3 95123 09760
28	1 31383 29600	2 33570 30400
30	1 42210 25280	2 45248 81920
35	1 40142 18240	–
36	–	2 72498 68800
40	1 22624 40960	40874 80320
42	2 04374 01600	1 16785 15200
48	–	2 04374 01600
56	–	1 75177 72800
60	61312 20480	3 26998 42560
70	1 40142 18240	–
84	–	1 16785 15200
120	–	81749 60640

Table 1. The number of elements of each possible order and of either parity in the edge group G_e.

Order	Even	Odd
1	1	-
2	10395	11424
3	3 55994	-
4	11 22660	3 17520
5	1 08864	-
6	43 52670	45 06432
7	41 99040	-
8	-	110 22480
9	23 40576	-
10	-	9 79776
12	71 44200	58 06080
15	47 90016	-
18	73 48320	87 09120
21	83 98080	-
30	-	78 38208
36	-	48 98880
45	39 19104	-

Table 2. The number of elements of each possible order and of either parity in the corner group G_c.

Order	Even-Even	Odd-Odd
1	1	–
2	8 40043 37407	8 69072 11776
3	3389 45406 22394	–
4	3 13917 01587 41760	1 20778 68714 02496
5	13352 81725 14624	–
6	82 50599 51264 65478	58 11506 41722 90048
7	15324 55171 48800	–
8	40 62074 72232 03840	254 37789 17587 35360
9	55 33375 23984 28896	–
10	8 15090 88984 43968	57 09998 89379 08224
11	4 45906 94400	–
12	904 27148 15102 89920	1425 96134 59452 64128
14	23 23166 67041 99040	6670 76788 22400
15	14 38547 13332 09856	–
16	34 73508 07609 34400	115 99680 55099 39200
18	408 59354 22578 80352	112 02920 69002 44480
20	127 77257 81519 30240	70 95966 75029 97504
21	39 33715 15593 33120	–
22	92708 51272 70400	–
24	745 62219 70012 56960	2548 31032 27949 87520
28	87 88252 48256 24000	9 53723 60810 49600
30	881 75820 64126 30272	1151 52600 25541 09952
33	15 87401 96622 33600	–
35	65 52621 89125 63200	–
36	738 09679 39107 84000	2448 10580 40623 92320
40	19 13181 13360 28160	1117 12656 89183 84640
42	658 12370 95588 03200	79 07606 72722 94400
44	100 12037 79502 08000	–
45	197 32944 16597 27104	–
48	363 36261 25172 73600	548 13503 48931 07200
55	4 85432 13551 61600	–
56	150 53813 97037 05600	520 66716 71427 07200
60	1686 14412 23105 74080	2915 38057 05823 51872
63	264 37143 37053 08160	–
66	404 05117 52503 29600	–
70	347 76966 98194 32960	5 72116 44542 97600

(cont.)

Table 3. The number of elements of each possible order in the cube group G_o, for even-even and for odd-odd parities of the edge and corner parts.

(cont.)

72	586 62198 11399 27040	1094 47067 91997 44000
77	187 23810 94133 76000	–
80	3 33734 59316 73600	10 01203 77950 20800
84	1227 16200 43026 43200	470 56381 43754 24000
90	1209 83873 68382 67648	715 23031 47791 15520
99	104 36790 91359 74400	–
105	232 82441 94233 54880	–
110	4 85432 13551 61600	–
112	128 72620 02216 96000	–
120	496 78611 24228 71040	1450 25789 90402 76480
126	273 13196 67759 51360	152 56438 54479 36000
132	637 12967 78649 60000	–
140	200 24075 59004 16000	22 88465 78171 90400
144	297 02378 79189 50400	417 16824 14592 00000
154	187 23810 94133 76000	–
165	213 59013 96271 10400	–
168	612 16459 66098 43200	438 10464 26566 65600
180	855 96777 36529 92000	1597 92123 20853 19680
198	759 70129 20827 90400	–
210	1190 48245 62352 12800	148 75027 58117 37600
231	374 47621 88267 52000	–
240	146 84322 09936 38400	260 31298 26705 40800
252	127 29590 91081 21600	562 58117 13392 64000
280	51 49048 00886 78400	17 16349 33628 92800
315	99 30987 96525 15840	–
330	213 59013 96271 10400	–
336	257 45240 04433 92000	–
360	404 15259 23256 72960	166 86729 65836 80000
420	675 09740 56071 16800	286 05822 27148 80000
462	374 47621 88267 52000	–
495	174 75556 87858 17600	–
504	–	238 38185 22624 00000
630	395 38014 01624 16640	–
720	120 14445 35402 49600	–
840	102 98096 01773 56800	137 30794 69031 42400
990	174 75556 87858 17600	–
1260	51 49048 00886 78400	–

Table 3 (continued).

Order	Number of elements	Order	Number of elements
1	1	77	187 23810 94133 76000
2	17 09115 49183	80	13 34938 37266 94400
3	3389 45406 22394	84	1697 72581 86780 67200
4	4 34695 70301 44256	90	1925 06905 16173 83168
5	13352 81725 14624	99	104 36790 91359 74400
6	140 62105 92987 55526	105	232 82441 94233 54880
7	15324 55171 48800	110	4 85432 13551 61600
8	294 99863 89819 39200	112	128 72620 02216 96000
9	55 33375 23984 28896	120	1947 04401 14631 47520
10	65 25089 78363 52192	126	425 69635 22238 87360
11	4 45906 94400	132	637 12967 78549 60000
12	2330 23282 74555 54048	140	223 12541 37176 06400
14	23 29837 43830 21440	144	714 19202 93781 50400
15	14 38547 13332 09856	154	187 23810 94133 76000
16	150 73188 62708 73600	165	213 59013 96271 10400
18	520 62284 91581 24832	168	1050 26923 92665 08800
20	198 73224 56649 27744	180	2453 88900 57383 11680
21	39 33715 15593 33120	198	759 70129 20827 90400
22	92708 51272 70400	210	1339 23273 20469 50400
24	3293 93251 97962 44480	231	374 47621 88267 52000
28	97 41976 09076 73600	240	407 15620 36641 79200
30	2033 28420 89667 40224	252	689 87708 04473 85600
33	15 87401 96622 33600	280	68 65397 34515 71200
35	65 52621 89125 63200	315	99 30987 96525 15840
36	3186 20259 79731 76320	330	213 59013 96271 10400
40	1136 25838 02544 12800	336	257 45240 04433 92000
42	737 19977 68310 97600	360	571 01988 89093 52960
44	100 12037 79502 08000	420	961 15562 83219 96800
45	197 32944 16597 27104	462	374 47621 88267 52000
48	911 49764 74103 80800	495	174 75556 87858 17600
55	4 85432 13551 61600	504	238 38185 22624 00000
56	671 20530 68464 12800	630	395 38014 01624 16640
60	4601 52469 28929 25952	720	120 14445 35402 49600
63	264 37143 37053 08160	840	240 28890 70804 99200
66	404 05117 52503 29600	990	174 75556 87858 17600
70	353 49083 42737 30560	1260	51 49048 00886 78400
72	1681 09266 03396 71040		

Table 4. The number of elements of each possible order in the group G_0.