# WARING'S CONJECTURE AND A CERTAIN DIOPHANTINE EQUATION

*Mike Mudge starts a new series of mathematical puzzles.*

*PCW* has always prided itself on its ability to provide answers to that perennial question, 'what can I use my computer for?' Most of the answers, for hobbyist or enthusiast, have been of a recreational or educational nature. There is however an area of serious research which is open to (and well within the capabilities of) the mathematically-minded computer hobbyist. I refer to the field of Number Theory. Given that its subject matter — the natural numbers — is infinite, it's not surprising that there remain huge numbers of unsolved problems; moreover, the scope for discovering new problems of interest is limitless.

From this issue on, we will be publishing a monthly column by Mike Mudge BSc FIMA FBCS of the University of Aston, in which he sets problems in Number Theory (with explanatory background) and awards a monthly prize of £10 for the best submission. These problems will not be simple puzzles but genuine research projects in Number Theory which are capable of investigation using a personal computer; who knows, we may even arrive at some important results? Don't be discouraged from trying them if you're not a professional mathematician; if you've enjoyed our Leisure Lines puzzles or the Manhunt Competition you might find that you're a budding number theorist already! — *Dick Pountain.*

## Waring's conjecture

The integers consist of 0, ±1, ±2, ±3, . . . eg, +1234 or -201379. When k denotes a positive integer, the product of k-factors each equal to x is written $x^k$, eg, $7^3 = 7.7.7 = 343$.

A Diophantine equation is one which is to be solved using integers only, the first writer to study such equations being Diophantus of Alexandria (c AD 250). For example $x^2 + y^2 = z^2$ regarded as a Diophantine equation has among its solutions x = 3, y = 4, z = 5 and x = 5, y = 12, z = 13 – each being the integer length sides of a Pythagorean (or right-angled) triangle.


*'There sir — Your flight number, departure time and the odds against being hijacked.'*

In 1770, in a text entitled *Meditationes Algebraicae*, the mathematician Edward Waring wrote (in Latin): 'Every positive integer can be expressed as the sum of at most, g(k), $k^{th}$ powers of the positive integers, where g(k) depends only on k, not on the number being represented.'

**Special Cases k = 2,3,6.**

It was proved by Lagrange in 1770 that every positive integer can be expressed as the sum of at most *four squares.* Other theoretical results to date include:

i. every positive integer can be expressed as the sum of at most *nine cubes.*

ii. every positive integer can be expressed as the sum of at most 73 *sixth powers.*

## Problem

Given the Diophantine equation $x^3 + y^3 + 2z^3 = k$ where k is a known positive integer, what are the (integer) solutions for x, y and z?

## Historical note

In 1969 M Lal, W Russell and W J Blundon (*Math Comp* Vol 23) reported a calculation which was originally programmed in Fortran and subsequently in assembler (showing an acceleration factor of 15x) for an IBM 1620; after 1000 hours at low priority they had considered $-10^5 < x,y,z < 10^5$ and all k between 1 and 999. Their computation revealed the results

$$(-133)^3 + (-46)^3 + 2(107)^3 = 113$$
$$(-602)^3 + (450)^3 + 2(309)^3 = 190$$
$$(-79)^3 + (126)^3 + 2(-91)^3 = 195$$

omitted by the previous writer, Makowski, in 1959.

However they failed to find any values of x,y,z corresponding to the following 19 k values less than 1000:

| | | | | |
|---|---|---|---|---|
| 76 | 148 | 183 | 230 | 253 |
| 356 | 418 | 428 | 445 | 482 |
| 491 | 519 | 580 | 671 | 734 |
| 788 | 923 | 931 | 967 | |

Submit a program which generates these numbers and attempt to eliminate some of them by an extended x,y,z search or otherwise. Alternatively extend the k-range, hence possibly adding to the above list.

All submissions should include program listings, hardware description, run times and output; they will be judged for accuracy, originality and efficiency (not necessarily in that order) and I shall award a suitable prize to the 'best' entry.

Submissions to: M R Mudge BSc FIMA FBCS, Room 560/A, Department of Mathematics, The University of Aston in Birmingham, Gosta Green, Birmingham B4 7ET.

*Note:* Submissions will only be returned if suitable stamped addressed envelopes are included.

# 1983 PCW SHOW PREVIEW

**Ulster Management Centre** Stand No 302
Manor House
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As well as a range of management training materials, Ulster Management Centre will demonstrate a game generator for the BBC, Apple, IBM, Sirius and Spectrum. Also on show will be a veterinary practice program on the Sirius, special service systems and a welfare benefits program on the Epson HX-20 — £850 complete with the computer.

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## NUMBERS COUNT

# ABUNDANT, DEFICIENT AND PERFECT NUMBERS... ALIQUOT SEQUENCES.

*New readers start here. The topics dealt with in this column attempt to reach the frontiers of knowledge in number theory with the minimal background information. The problems posed therefore have no complete solution known to the author, and readers are encouraged to submit their attempts at solution, however incomplete they may seem.*

A proper divisor of an integer n is any positive integer divisor of n except n itself. f(n) denotes the sum of the proper divisors of n, and $f_k(n)$ denotes the sum of the $k^{th}$ powers of these divisors — eg, f(6) = 1+2+3=6, f(15) = 1+3+5=9.

The divisors of an integer n consist of the proper divisors of n, defined above, together with n itself. $\sigma(n)$ denotes the sum of the divisors of n, and $\sigma_k(n)$ denotes the sum of the $k^{th}$ powers of these divisors. Thus $\sigma(n) = f(n) + n$, while $\sigma_k(n) = f_k(n) + n^k$.

n is Perfect if and only if $\sigma(n) + 2n$, *viz*, f(n) = n.

n is Abundant if and if only $\sigma(n) > 2n$.

nn is Deficient if and only if $\sigma(n) < 2n$.

eg, 6, 28, and 496 are perfect since:

1+2+3+6 = 2.6 = 12; 1+2+4+7+14+28 = 2.28 = 56;

1+2+4+8+16+31+62+124+248 = 2.248 = 496.

Since some numbers are known to be abundant and some deficient, it is natural to ask what happens when we iterate the function f(n) to produce an Aliquot Sequence $\{f^m(n)\}$ m = 1,2, . . . where by iteration we mean repeated application of the function, eg $f^3(15) = f(f(f(15))) = f(f(9)) = f(4) = 3$.

Now E Catalan Bull, Soc Math France 16 (1887-88) pp128-129, conjectured that the iteration is either periodic or stops at the number 1.

There now exists a heuristic argument together with much experimental evidence to suggest that some sequences, perhaps almost all of those with n even, are of infinite length.

P Poulet has calculated that for n=936 we obtain the sequence 936, 1794, 2238, 2250, . . . 74, 40, 50, 43, 1 containing 189 terms, the greatest of which has 15 digits.

The smallest n for which the behaviour was in doubt was 138 but D H Lehmer eventually showed that, after reaching a maximum of $f^{117}(138) = 179931895322 = 2.61.929.1587569$, the sequence terminated at $f^{177}(138) = 1$.

The next value for which there continues to be real doubt is 276 $f^{469}(276)$ = 149384846598254844243905695992651412919855640 reported to 3rd Conf Numerical Math Winnipeg 1973 by R K Guy, D H Lehmer, J L Selfridge and M C Wunderlich.

# Problem

Submit a program, or suite of programs, to determine if a given integer is perfect, abundant or deficient . . . check that there are 23 odd abundant numbers less than 10,000 . . . use the same routine to iterate either the f(n) or $\sigma$ (n) function and display the resulting sequences in the most useful manner to shed light upon the Catalan Conjecture.

All submissions should include program listings, hardware descriptions, run times and output; they will be judged for accuracy, originality and efficiency (not necessarily in that order). A suitable prize will be awarded to the 'best' entry received.

Entries, to arrive by 1 December, to: Mr M R Mudge, BSc FIMA FBCS, Room 560/A, Department of Mathematics, The University of Aston in Birmingham, Gosta Green, Birmingham B4 7ET.

*Note: Submissions will only be returned if suitable stamped addressed envelopes are included.*